Schwartz 2.1

Begin with $x \to x + v_g t$

By method of perturbation, let

$t \to t + \delta t$, then

$$(x + v_g t)^2 - x^2 = (t + \delta t)^2 - t^2$$

$$v_g^2 t^2 + 2 v_g x t = (\delta t)^2 + 2t(\delta t)$$

With $v \equiv v_g$, to order $O(v^2)$, it is

$$v^2 t^2 + 2 v x t = (\delta t)^2 + 2t(\delta t),$$

$$2 v x t + O(v^2) = (\delta t)^2 + 2t(\delta t)$$

If δt is linear in x, t, then $(\delta t)^2$ is second order in x, t, but there are no $O(x^2)$ or $O(t^2)$ terms here, so let's throw away $(\delta t)^2$, it gives

$$2t(\delta t) = 2 v x t + O(v^2) + O(\delta t^2)$$

$$\delta t = v x,$$

thus we have $x \to x + v_g t$

$t \to t + v x.$

v gets the interpretation of "adjusted velocity", of course, so we have

$$x \to x + v_g t$$
$$t \to t + V(v_g) x.$$

$t^2 - x^2$ being ~~conserved~~ Lorentz invariant demands the following constraint on $V(v_g)$:

$$(x + v_g t)^2 - x^2 = (t + Vx)^2 - t^2$$

$$2x v_g t + v_g^2 t^2 = 2xVt + V^2 x^2,$$

To $O(V^2)$ it gives

$$2xVt + O(V^2) = 2x v_g t + v_g^2 t^2$$

$$2xVt \approx 2x v_g t + v_g^2 t^2$$

$$\boxed{V(v_g) \approx v_g + \frac{1}{2}\frac{v_g^2 t}{x}}$$

Plugging it back in gives

$$x \to x + v_g t$$
$$t \to t + v_g x + \frac{1}{2} v_g^2 t$$

In the exact formula, $t \to \frac{t + vx}{\sqrt{1-v^2}}$

$\frac{1}{\sqrt{1-v^2}} \approx 1 + \frac{v^2}{2} + \frac{3v^4}{8}$, so

$$t \approx t + vx\left(1 + \frac{v^2}{2}\right) + \cdots$$

$$\approx \boxed{t + vx + \frac{v^2}{2}t} + \cdots$$

This agrees with our result of $t \to t + v_g x + \frac{1}{2} v_g^2 t$,

with replacement $v \to v_g$.

Applying this method to $x \to x'$, we would get

$$x \to x + v_g t + \frac{1}{2} v_g^2 x + \cdots,$$

because of symmetry in exchange $x \leftrightarrow t$.

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